University of California, Berkeley
Physics 110B, Fall 2004 (*Strovink*)
PROBLEM SET 10
Due at 5 PM on Wednesday, November 3, 2004

Problems 48-52 offer practice with complex EM fields and with the Jones vectors used to represent fully polarized light.

48.
Consider a medium with uniform fixed dielectric constant ϵ, permeability μ, and volume conductivity σ.
(a.)
Taking the curl of the two Maxwell equations which themselves involve the curl, and using Ohm's law ($\vec{J} = \sigma\vec{E}$) and the two other Maxwell equations where appropriate, derive the wave equations

$$\Big(\nabla^2 - \frac{\partial^2}{v^2\,\partial t^2}\Big)\vec{B} = \sigma\mu\frac{\partial\vec{B}}{\partial t}$$
$$\Big(\nabla^2 - \frac{\partial^2}{v^2\,\partial t^2}\Big)\vec{E} = \sigma\mu\frac{\partial\vec{E}}{\partial t} + \frac{1}{\epsilon}\nabla\rho_f\ ,$$

where the phase velocity[2] $v^2 \equiv \frac{1}{\epsilon\mu}$ and ρ_f is the volume free charge density.
(b.)
In the wave equation for $\vec{B}$ derived in (a.), substitute

$$\vec{B}(\vec{r},t) = \mathrm{Re}\Big(\tilde{\vec{B}}\exp\big(i(\tilde{\vec{k}}\cdot\vec{r} - \omega t)\big)\Big)\ ,$$

where $\tilde{\vec{B}}$ and $\tilde{\vec{k}}$ are complex (vector) constants. Show that

$$\frac{\tilde{k}^2}{\mu\omega^2} = \epsilon(1 + i\beta)\ ,$$

where $\beta \equiv \frac{\sigma}{\epsilon\omega}$.
(c.)
Following Griffiths' notation, write $\tilde{k} \equiv k + i\kappa$, where k and κ are real. Show that

$$\frac{k}{\omega/v} = \sqrt{\frac{\sqrt{1+\beta^2}+1}{2}}$$
$$\frac{\kappa}{\omega/v} = \sqrt{\frac{\sqrt{1+\beta^2}-1}{2}}$$

are solutions to the equation that is the result of part (b.).

(d.)
κ^{-1}, the inverse of the imaginary part of $\tilde{k}$, is called the *skin depth*. Show that the skin depth approaches

$$\sqrt{\frac{2}{\mu\sigma\omega}}\ \text{ when } \beta \gg 1\ \text{ (good conductor)}$$
$$\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}\ \text{ when } \beta \ll 1\ \text{ (poor conductor)}\ .$$

49.
Please refer to the notation and results of the previous problem.
(a.)
At normal incidence at the interface between two dissimilar materials 1 and 2, the (complex) electric field amplitude reflected back into material 1 is expressed as a (complex) ratio $\tilde{\mathcal{R}}$ to the (complex) incident amplitude. By matching boundary conditions for the electric and magnetic fields, $\tilde{\mathcal{R}}$ is routinely found to be given by the standard result

$$\tilde{\mathcal{R}} = \frac{\tilde{Z}_1^{-1} - \tilde{Z}_2^{-1}}{\tilde{Z}_1^{-1} + \tilde{Z}_2^{-1}}\ ,$$

where

$$\tilde{Z}^{-1} \equiv \frac{\tilde{k}}{\mu\omega}\ ,$$

the ratio of $\tilde{H}$ to $\tilde{E}$, is the medium's (complex) *admittance*. Consider the case in which material 1 is an insulator and material 2 is a conductor.

If material 2 is an *excellent* conductor ($\beta \gg 1$), show that $\tilde{\mathcal{R}} \to -1$ regardless of the (finite) values taken by $\epsilon_{1,2}$ and $\mu_{1,2}$. Therefore *metals are shiny*.

(b.)
Suppose instead that material 2 is a *poor* conductor ($\beta \ll 1$) (otherwise all the conditions of part (a.) apply). Suppose further that, if both materials had zero conductivity, they would have equal admittance ($\sqrt{\epsilon_1/\mu_1} = \sqrt{\epsilon_2/\mu_2}$). Show that $\tilde{\mathcal{R}} \to -i\beta/4$.
(c.)
In a relatively more microscopic and detailed treatment, one assumes that N valence electrons per m^3 having charge $-e$ and mass m move in a potential well with effective spring constant $m\omega_0^2$ and damping coefficient γm. One defines the *complex dielectric constant* $\tilde{\epsilon}$ via

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 \equiv \frac{\tilde{P}}{\epsilon_0 \tilde{E}} ,$$

where $\tilde{P}$ is the complex polarization, defined analogously to the complex electric and magnetic fields in the previous problem. For not-too-dense media in which the electric field felt by the electron is approximately the same as the average field, it is straightforward to solve the force equation for these oscillating electrons and determine the complex polarization $\tilde{P}$ they create. One obtains

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} ,$$

where the *plasma frequency*^{2} is

$$\omega_p^2 \equiv \frac{Ne^2}{m\epsilon_0} .$$

The complex dielectric constant includes the effects of all electrons (free and bound). Considering the result of part (b.) of the previous problem, the complex dielectric constant is related to the ordinary dielectric constant ϵ (which includes the effects only of bound electrons) by

$$\tilde{\epsilon} = \epsilon(1 + i\beta) = \frac{\tilde{k}^2}{\mu\omega^2} .$$

Represent a *good conductor* by $\omega_0 = 0$ (unbound) and $\gamma \gg \omega$ (overdamped). Using these results, show that the conductivity σ is approximately

$$\sigma \approx \frac{\epsilon_0 \omega_p^2}{\gamma} ,$$

i.e. measuring the low-frequency conductivity is a simple way to determine the damping coefficient.
(d.)
Represent the *ionosphere* by $\omega_0 = 0$ (unbound), and $\gamma \ll \omega$ (underdamped). Specialize to AM radio waves, for which $\omega < \omega_p$. Show that $|\tilde{\mathcal{R}}| \approx 1$, *i.e.* that AM radio waves are nearly fully reflected by the ionosphere. (At dusk, the ionosphere drops to sufficiently low altitude that reflection off it enables AM stations hundreds of miles away to be received.)

50.
Griffiths Problem 9.11.

51. Jones vectors.
For a plane transverse wave propagating in the $\hat{z}$ direction through a (not necessarily insulating) material with constant ϵ and μ, a (co)sinusoidal solution is represented by

$$\vec{E}(\vec{r}, t) = \mathrm{Re}\Big(\vec{E}_0(x, y)\, e^{i(\tilde{k}z - \omega t)}\Big)$$
$$\vec{H}(\vec{r}, t) = \mathrm{Re}\Big(\vec{H}_0(x, y)\, e^{i(\tilde{k}z - \omega t)}\Big) ,$$

where $\tilde{k}$ is the (not necessarily real) "wave vector" – here a scalar because we know it is directed along $\hat{z}$. Faraday's law causes $\vec{H}_0$ to be completely determined by $\vec{E}_0$:

$$\begin{aligned} \vec{H}_0 &\equiv \tilde{Z}^{-1} \hat{z} \times \vec{E}_0 \\ &= \frac{\tilde{k}}{\mu\omega} \hat{z} \times \vec{E}_0 , \end{aligned}$$

so we focus on $\vec{E}_0$ as the sole independent variable. For a transverse wave $\vec{E}_0$ has no z component. Here we assume that the phase relationship between E_{0x} and E_{0y} is *fixed* – the wave is *fully polarized*. Then $\vec{E}_0$ is a complex transverse vector, completely specified by four components. In the Jones convention, all information carried by

$\vec{E}_0$ except for its magnitude is written as a 2×1 column vector with the x component on top:

$$\begin{aligned} \vec{E}_0 &= \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} \\ &\equiv \frac{1}{\sqrt{|\alpha^2| + |\beta^2|}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |\vec{E}_0| \\ &\equiv \vec{J}\,|\vec{E}_0| \;, \end{aligned}$$

where $\vec{J}$ is the *Jones vector*. Jones vectors are defined only within an overall phase (because the absolute phase of an optical-frequency EM wave can't conveniently be measured); therefore one has the freedom to set α equal to unity (unless it vanishes, in which case β is set to unity). The above form involving the complex constants α and β is a general Jones vector, corresponding to elliptical polarization. More common Jones vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \;\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad ,$$

corresponding, respectively, to linear x, linear y, RH circular, and LH circular polarization.

(a.)

At $z = 0$, show (counterintuitively!) that the electric field vector for RH polarized light precesses *clockwise* around $\hat{z}$, *i.e.* it precesses according to the LH rule.

(b.)

Suppose that a particular state of elliptical polarization has nonvanishing x and y electric field components. Then, within an arbitrary overall phase, it may be represented by the Jones vector

$$\vec{J}_1 = \frac{1}{\sqrt{1 + |\beta|^2}} \begin{pmatrix} 1 \\ \beta \end{pmatrix} \;,$$

where β is a complex constant. You wish to characterize this state of polarization as "RH elliptical" or "LH elliptical", depending on whether (at $z = 0$) the electric field vector precesses clockwise or counterclockwise around $\hat{z}$. What property of β would you use to decide whether this state is RH or LH elliptical?

(c.)

For the conditions of part (b.), decompose $\vec{J}_1$ into a linear sum (with real coefficients) of a wave with linear polarization plus a wave with RH circular polarization. Perform this same task with "RH" replaced by "LH". If you are successful in both tasks, you might wonder whether there really exists a unique association of RH or LH behavior with $\vec{J}_1$. Would this concern invalidate your answer to (b.)?

52. Irradiance and Jones vectors.

Consider two transverse plane waves A and B which move in vacuum and are combined together (*i.e.* by a Michelson interferometer). The beams have complex electric fields

$$\begin{aligned} \begin{pmatrix} E^A_{0x} \\ E^A_{0y} \end{pmatrix} &= \frac{|\vec{E}^A_0|}{\sqrt{|\alpha|^2 + |\beta|^2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{pmatrix} E^B_{0x} \\ E^B_{0y} \end{pmatrix} &= \frac{|\vec{E}^B_0|}{\sqrt{|\gamma|^2 + |\delta|^2}} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \;. \end{aligned}$$

Express the combined irradiance

$$I_{A+B} \equiv \langle \vec{S}_{A+B} \cdot \hat{z} \rangle \;,$$

where $\vec{S}$ is the Poynting vector and $\langle\;\rangle$ is a time average, as a function of the complex constants α, β, γ, δ, and the uncombined irradiances I_A and I_B of the individual beams.